Quantum Schur–Weyl duality and link invariants

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1Happy to hear your questions or comments!
Quantum Schur–Weyl duality and link invariants

Outline

1 Quantum Schur–Weyl duality

2 Link invariants

3 An example
Representation categories of Lie algebras

\( g \): a Lie algebra over \( \mathbb{F} \)

\( \text{Rep} \): the category of (finite dimensional) representations

- \( \mathbb{F} \in \text{Rep} : \)
  \[
  xr = 0 \quad \forall r \in \mathbb{F}, x \in g .
  \]

- \( M, N \in \text{Rep} \Rightarrow M \otimes N \in \text{Rep} : \)
  \[
  x(m \otimes n) = xm \otimes n + m \otimes xn \quad \forall m \in M, n \in N .
  \]

- \( M \in \text{Rep} \Rightarrow M^* = \text{Hom}_{\mathbb{F}}(M, \mathbb{F}) \in \text{Rep} : \)
  \[
  xf = -f(x \cdot) \quad \forall f \in M^* .
  \]

“−” implies:
\( M \otimes M^* \rightarrow \mathbb{F} \) is a morphism in \( \text{Rep} \), i.e., a \( g \)-module map
Associative algebras are nicer than Lie algebras...

\[ U = U(\mathfrak{g}) : \text{the universal enveloping algebra of } \mathfrak{g} \text{ (associative } + 1), \]
\[ U = T(\mathfrak{g})/(xy - yx - [x, y]) ; \text{generated by } \{x\}_{x \in \mathfrak{g}} \text{ with relations} \]
\[ xy - yx = [x, y] \ \forall x, y \in \mathfrak{g}. \]

⇒ Rep = the category of (finite-dimensional) U-modules

We have algebra maps \( \varepsilon : U \to \mathbb{F}, \Delta : U \to U \otimes U, S : U \to U \)
defined on the generators \( \{x\}_{x \in \mathfrak{g}} \) by
\[ \varepsilon(x) = 0, \quad \Delta(x) = x \otimes 1 + 1 \otimes x, \quad S(x) = -x \]
such that:
\[ \mathbb{F} \in \text{Rep} : ur = \varepsilon(u)r \ \forall u \in U \]
\[ M \otimes N \in \text{Rep} : u(m \otimes n) = \Delta(u)(m \otimes n) \]
\[ M^* \in \text{Rep} : uf = f(S(u) \cdot) \]
Algebras with additional structure maps $\varepsilon, \Delta, S$ as above satisfying certain axioms are called **Hopf algebras**. Their representation categories are **rigid monoidal** categories (“they have duals and tensor products”).

⇒ $\mathcal{U}$ is a Hopf algebra

∀ vector spaces $V, W$: $\tau_{V,W}: V \otimes W \rightarrow W \otimes V$, $v \otimes w \mapsto w \otimes v$.

⇒ $\Delta = \tau_{\mathcal{U},\mathcal{U}} \circ \Delta$ (⇔: $\mathcal{U}$ is cocommutative)

⇔ Rep is **symmetric monoidal** with the symmetric braiding $\tau$: ∀ $M, N \in$ Rep, $\tau_{M,N}$ gives an isomorphism in Rep and $\tau^2 = \text{id}$.

**Caution**: Cocommutative Hopf algebras / symmetric monoidal categories are a special case!
Endomorphisms of tensor powers

We fix a module $M \in \text{Rep}$ and $n \geq 1$.

$\Rightarrow M^\otimes n \in \text{Rep}$ and we have an algebra map
$\phi (= \Delta^{n-1}) : \mathcal{U} \rightarrow \text{End}(M^\otimes n)$

$s_i := \text{id}^\otimes(i-1) \otimes \tau_{M,M} \otimes \text{id}^\otimes(n-i-1) : M^\otimes n \rightarrow M^\otimes n$ for $1 \leq i < n$

- $\psi : S_n \rightarrow \text{GL}(M^\otimes n), (i \ i + 1) \mapsto s_i$ defines a group homo.
  $(s_i^2 = \text{id}, s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1, s_i s_j = s_j s_i$ if $|i - j| > 1)$

- So we have an algebra map $\psi : \mathbb{F}[S_n] \rightarrow \text{End}(M^\otimes n)$.

- For all $u \in \mathcal{U}$ and all $i$: $\phi(u), \psi(s_i)$ commute!

$\Rightarrow \phi(\mathcal{U}), \psi(\mathbb{F}[S_n])$ are commuting algebras in $\text{End}(M^\otimes n)$

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$\mathbb{F}[G]$ is the algebra gen. by $\{e_g\}_{g \in G}$ with rel.s $e_g e_h = e_{gh}$. 

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Let’s specialize to $\mathfrak{gl}_d(\mathbb{C})$

Let us specialize $\mathbb{F} = \mathbb{C}$, $\mathfrak{g} = \mathfrak{gl}_d(\mathbb{C})$, $M = \mathbb{C}^d$ for $d \geq 1$.

**Schur–Weyl duality**

$\phi(\mathcal{U}(\mathfrak{gl}_d)), \psi(\mathbb{C}[S_n])$ are (full!) commutators of each other in $\text{End}((\mathbb{C}^d)^{\otimes n})$.

As a corollary, $(\mathbb{C}^d)^{\otimes n} = \bigoplus_\lambda V_\lambda \otimes W_\lambda$ for pairwise non-isomorphic irreducible $\mathfrak{gl}_d$-modules $V_\lambda$ / $S_n$-modules $W_\lambda$.

More concretely, $\{\lambda\}$ can be taken to be the set of partitions of $n$ with at most $d$ parts. (Equivalently, partitions of $n$ with all parts being at most $d$.)
Quantization

For \( q \in \mathbb{C} \setminus \{0, 1\} \), \( \mathcal{U}_q = \mathcal{U}_q(gl_d) \) is a Hopf algebra deformation of \( \mathcal{U} = \mathcal{U}(gl_d) \) such that \( \mathcal{U}_q \to \mathcal{U} \) as \( q \to 1 \).

\( \mathcal{U}_q \) (still) has \( \mathbb{C}^d \) as a natural standard module.

The representation category \( \text{Rep}_q \) is rigid monoidal, but not symmetric anymore. \( S_n \) does not act on \( (\mathbb{C}^d)^{\otimes n} \).

There is still a braiding \( c_{M,N} : M \otimes N \to N \otimes M \) for \( M, N \in \text{Rep}_q \) with \( c^2 \neq \text{id} \) generally.

\( \text{Rep}_q \) is (still) a ribbon category: it has tensor products, duals, a braiding and twists, and they are compatible.
Quantum Schur–Weyl duality

\[ S_n = \text{group generated by } s_1, \ldots, s_{n-1} \text{ and relations:} \]
\[ s_i^2 = 1, \quad s_i s_j s_i = s_j s_i s_j \text{ if } |i - j| = 1, \quad s_i s_j = s_j s_i \text{ if } |i - j| > 1 \]

braid relations

\[ U(\mathfrak{gl}_d) \xrightarrow{\phi} \text{End}((\mathbb{C}^d)^\otimes n) \xleftarrow{\psi} \mathbb{C}[S_n] \]

double centralizer

braid group \( B_{n} = \text{group generated by } \sigma_1, \ldots, \sigma_{n-1} \) with braid relations

Hecke algebra \( H_{q,n} = \mathbb{C}\text{-algebra generated by } T_1, \ldots, T_{n-1} \) with braid relations and \((T_i + q)(T_i - q^{-1}) = 1\)
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3 An example
Braids and links – Alexander

(Oriented) link := finite collection of smoothly embedded (oriented) circles in 3-space

Alexander’s theorem

(Oriented) links are closures of (oriented) braids.
We fix $M \in \text{Rep}_q$.

- $\mathbb{C}[\text{Br}_n] \to \text{End}(M \otimes^n)$, braid $\mapsto$ endomorphism
- closing the braid $\leftrightarrow$ taking the trace

**Reshetikhin–Turaev**

The ribbon category $\text{Rep}_q$ yields link invariants in this way.
Braids and links – Markov

\[ \frac{\{\text{links}\}}{\text{isotopy}} \leftrightarrow \frac{\{\text{braids}\}}{\text{conjugations, Markov moves}} \]

(*)
Link invariants from Hecke algebras

knots $\rightarrow$ braids $\rightarrow \bigcup_{n \geq 1} \mathbb{C}[\text{Br}_n] \rightarrow \mathcal{H}_q := \bigcup_{n \geq 1} \mathcal{H}_{q,n}$

A linear map $\text{Tr} : \mathcal{H}_q \rightarrow \mathbb{C}$ is called normalized Markov trace with parameter $z \in \mathbb{C}$

$\Leftrightarrow \text{Tr}(1) = 1, \quad \text{Tr}(ab) = \text{Tr}(ba), \quad \text{Tr}(M(b)) = z \text{Tr}(b)$

for all $a, b \in \mathcal{H}_q$, where $M(b)$ is the modification of $b$ according to the Markov move.

Ocneanu

For all $q, z$, there is a unique normalized Markov trace.

Jones

Every normalized Markov trace yields an invariant for oriented links. Ocneanu’s trace yields the two-parameter HOMFLYPT polynomial.
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The Temperley–Lieb algebra

For $d = 2$, the image of $\mathbb{C}[\text{Br}_n] \rightarrow \text{End}((\mathbb{C}^2)^\otimes n)$ is...

Temperley–Lieb algebra $\text{TL}_n(\delta)$ generated by $u_1, \ldots, u_{n-1}$ with the relations:

\[ u_i^2 = \delta u_i, \quad u_i u_j u_i = u_i \text{ if } |i - j| = 1, \quad u_i u_j = u_j u_i \text{ if } |i - j| > 1. \]

Graphically, $u_i$ corresponds to $\cdots \bigcirc \bigcirc \cdots$ and composition corresponds to stacking diagrams ("crossingless matchings"), where circles are evaluated to $\delta$.

E.g.,

\[ u_i u_{i+1} u_i = \cdots \bigcirc \bigcirc \cdots = \cdots \bigcirc \bigcirc \cdots = u_i \]
Quantum Schur–Weyl duality and link invariants

An example

Braids and the Temperley–Lieb algebra

For any $\nu \in \mathbb{C}$, we have a group homomorphism $\eta: \text{Br}_n \to \text{TL}_n(\delta)$ sending $\sigma_i \mapsto \nu u_i + \nu^{-1}$, i.e.,

$$
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{braids.png}
\end{array}
\mapsto \nu \includegraphics[width=0.2\textwidth]{tangles.png} + \nu^{-1}
$$

for $\delta = -\nu^2 - \nu^{-2}$.

Pf.: By graphical calculus, e.g.,

$$
(\nu \includegraphics[width=0.2\textwidth]{braids.png} + \nu^{-1} | | ) (\nu^{-1} \includegraphics[width=0.2\textwidth]{braids.png} + \nu | | ) = \includegraphics[width=0.2\textwidth]{tangles.png} + \nu^2 \includegraphics[width=0.2\textwidth]{tangles.png} + \nu^{-2} \includegraphics[width=0.2\textwidth]{tangles.png} + | | =
$$

$$(\delta + \nu^2 + \nu^{-2}) \includegraphics[width=0.2\textwidth]{braids.png} + | | = | | \Rightarrow \eta \left( \includegraphics[width=0.2\textwidth]{braids.png} \right) = \nu^{-1} \includegraphics[width=0.2\textwidth]{tangles.png} + \nu | |
$$

$$
\eta \left( \includegraphics[width=0.2\textwidth]{braids.png} \right) = \eta \left( \nu^{-1} \includegraphics[width=0.2\textwidth]{braids.png} + \nu \includegraphics[width=0.2\textwidth]{tangles.png} \right) = \eta \left( \nu^{-1} \includegraphics[width=0.2\textwidth]{tangles.png} + \nu \includegraphics[width=0.2\textwidth]{tangles.png} \right) =
$$

$$
... = \eta \left( \includegraphics[width=0.2\textwidth]{braids.png} \right) ...
$$
Always trouble with the Markov move

**But:** Above assignment is **not** invariant under the Markov move!

Recall $\delta = -\nu^2 - \nu^{-2}$:

\[
\begin{align*}
\begin{array}{c}
\tikz[baseline=-.75ex] 
\draw (-.25,-.25) -- (-.25,.25) 
\draw (.25,-.25) -- (.25,.25) 
\end{array} & \mapsto \nu \begin{array}{c}
\tikz[baseline=-.75ex] 
\draw (-.25,-.25) -- (-.25,.25) 
\draw (.25,-.25) -- (.25,.25) 
\end{array} + \nu^{-1} | 0 = (\nu + \nu^{-1}\delta^2) | = -\nu^{-3} |, \\
\begin{array}{c}
\tikz[baseline=-.75ex] 
\draw (-.25,-.25) -- (-.25,.25) 
\draw (.25,-.25) -- (.25,.25) 
\end{array} & \mapsto \nu^{-1} \begin{array}{c}
\tikz[baseline=-.75ex] 
\draw (-.25,-.25) -- (-.25,.25) 
\draw (.25,-.25) -- (.25,.25) 
\end{array} + \nu | 0 = -\nu^3 |.
\end{align*}
\]

We obtain an assignment invariant under the Markov move by passing to **oriented** links and letting

\[
\begin{align*}
\begin{array}{c}
\tikz[baseline=-.75ex] 
\draw (-.25,-.25) -- (-.25,.25) 
\draw (.25,-.25) -- (.25,.25) 
\end{array} & \mapsto -\nu^\pm 3 \begin{array}{c}
\begin{array}{c}
\tikz[baseline=-.75ex] 
\draw (-.25,-.25) -- (-.25,.25) 
\draw (.25,-.25) -- (.25,.25) 
\end{array} + \nu^\mp 1 | |
\end{array} 
\end{align*}
\]

Now both \( \begin{array}{c}
\tikz[baseline=-.75ex] 
\draw (-.25,-.25) -- (-.25,.25) 
\draw (.25,-.25) -- (.25,.25) 
\end{array} \) and \( \begin{array}{c}
\tikz[baseline=-.75ex] 
\draw (-.25,-.25) -- (-.25,.25) 
\draw (.25,-.25) -- (.25,.25) 
\end{array} \) are mapped to \(| | \).
Link invariants from the Temperley–Lieb algebra

We define the trace $\text{Tr} : \text{TL}_n(\delta) \to \mathbb{C}$ by “closing the diagram”

$$\text{Tr}(d) = d$$

where each circle gets evaluated to $\delta$.

Let $q := -\nu^{-2}$. Recall $\delta = -\nu^2 - \nu^{-2} = q + q^{-1}$.

The Markov invariant assignment together with the trace map define an invariant $J$ for oriented links with normalization $J(\bigcirc) = \delta = q + q^{-1}$ and skein relation

$$q^2 J(\bigtriangledown) - q^{-2} J(\bigtriangledown) = (q - q^{-1}) J(\uparrow\uparrow).$$

This is the Jones polynomial (up to the normalization)!
There is a family of invariants \((P_n)_{n \geq 0}\) with skein relation
\[ q^n P_n( \underbrace{\quad \quad} ) - q^{-n} P_n( \underbrace{\quad \quad} ) = (q - q^{-1}) P_n( \uparrow \uparrow ) \]
and the normalization \(P_n( \bigcirc \bigcirc ) = \frac{q^n - q^{-n}}{q - q^{-1}}\).

\(P_0 = \text{Alexander polynomial, } P_1 \equiv 1, P_2 = \text{Jones polynomial, ...}\)

- All of these can be obtained from quantum groups, too.
- The HOMFLYPT polynomial is a 2-parameter generalization.
- The HOMFLYPT polynomial is not a complete invariant.
- Categorification \(\Rightarrow\) HOMFLYPT is the Euler characteristic of “Khovanov’s triply graded link homology”.
References


